Letter from A. M. Turing to I. J. Good referring to the manuscript of Probability and the Weighing of Evidence.

Oct '46

I have been reading your book, and will give more comments later, but for present would like to remark on the question of binomial distribution tending to normal. I know it is customary to deduce this from Stirling's formula, but it always seems to me that the 'real reason' is very much simpler. The following argument is much more general, and if you like you can deduce Stirling (with $\sqrt{2\pi}$ and all) from it.

Suppose $g(r) = a\prod_{s=1}^{r} h(s)$ (e.g. $g(r) = \binom{n}{r} p^r (1-p)^{n-r}$, $h(s) = \frac{p}{1-p}\cdot\frac{n-s+1}{s}$) (& $a = q^n$)

and $\nu(r) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(r-r_0)^2/2\sigma^2}$ (r_0 not nec. integral)

Then

$$\log\left(\frac{g(r_1)}{\nu(r_1)} \Big/ \frac{g(r_2)}{\nu(r_2)}\right) = \sum_{s=r_2+1}^{r_1}\left(\log h(s) + \frac{1}{\sigma^2}\left(s - r_0 - \tfrac{1}{2}\right)\right)$$

Now let r_0 be so chosen that $h(r_0+\frac{1}{2}) = 1$, and let

$h'(r_0+\frac{1}{2}) = -\frac{1}{\sigma^2}$, and suppose that

$\left|\frac{d^2}{ds^2}\log h(s)\right| < K$ provided $|s - r_0| < A$ say,

then $\left|\log\left(\frac{g(r_1)}{\nu(r_1)} \Big/ \frac{g(r_2)}{\nu(r_2)}\right)\right| \leq \frac{1}{2}K\sum_{r_2+1}^{r_1}(s-r_0)^2$ $|r_1 - r_0| < A$, $|r_2 - r_0| < A$

If r_2 is the nearest integer to r_0 we can write

$$\left|\log\left(\quad\right)\right| \leq \frac{1}{6}K\left(|r_1 - r_0| + \tfrac{3}{2}\right)^3 .$$

~~$\sigma^2 = \frac{r_0(n-r_0-1)}{n-1}$~~

$\sigma^2 = pq(n+1)$

Thus σ is not exactly equal to the S.D. of the original distribution.

In our particular case we have $h(s) = \frac{p}{1-p}\,\frac{n-s+1}{s}$, $r_0 = p(n+1) - \frac{1}{2}$,

and $\frac{d^2}{ds^2}\log h(s) = \frac{1}{s^2} - \frac{1}{(n-s+1)^2}$. We can take K to be

$\mathrm{Max}\left[\frac{4}{r_0^2}, \frac{4}{(n+1-r_0)^2}\right]$ if $A = \mathrm{Min}\left(\frac{r_0}{2}\text{ say}, \frac{n+1-r_0}{2}\right)$. This in effect means that the approximation $\nu(r)$ for $g(r)$ is good apart possibly from a constant factor so long as $s - r_0$ is small in comparison with r_0 and $n - r_0$. However ~~before~~ when $s - r_0$ is large enough to make this invalid both $g(r)$ and $\nu(r)$ are negligibly small. The constant factor must be 1 by the argument about the sum and the integral.

Yours

Prof

Donald has agreed to help, and I have now got the necessary gadgets for the [illegible].